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SPECTRUM OF LINE ARRANGEMENTS WITH MULTIPLICITIES: A COMBINATORIAL APPROACH VIA LATTICE POINT ENUMERATION

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Abstract. We introduce a combinatorial method for computing the spectrum of singularities using lattice point enumeration in regions determined by the defining equation. For line arrangements with multiplicities, our approach reveals the interplay between global and local contributions to the spectrum through explicit counting of lattice points. A series of transformations of these regions preserves the spectral data while simplifying the counting process. We highlight the case of $f(x, y) = x^{m_1}y^{m_2}$ as an illustrative example, where the method reduces the problem to counting lattice points on two line segments and naturally explains the role of $\gcd(m_1, m_2)$. This work provides a new perspective on the spectra of line arrangements with multiplicities, with potential applications to general non-isolated singularities.

1. Introduction

The spectrum of a hypersurface singularity, introduced by Steenbrink [2], is a fundamental invariant that encodes both the monodromy action and mixed Hodge structure on the vanishing cohomology. For isolated singularities, this invariant is well understood through various methods. In particular, when a singularity

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is non-degenerate with respect to its Newton boundary, the lattice points in the region bounded by the coordinate axes and the Newton boundary completely determine the spectrum. This method works effectively because, for isolated singularities, the Newton boundary intersects with the coordinate axes, ensuring that this region contains only finitely many lattice points.

However, for non-isolated singularities such as line arrangements with multiplicities, the situation becomes fundamentally different. In these cases, the Newton boundary typically does not intersect with the coordinate axes, and moreover, most such singularities are degenerate with respect to their Newton boundary. These structural differences mean that the classical approach of counting lattice points in Newton polygons cannot be directly applied, necessitating a new methodology.

Our key insight is that for line arrangements with multiplicities, the spectrum can be computed through explicit enumeration of lattice points in specific regions determined by the defining equation. These regions, built from generating vectors encoding degree and multiplicities, admit transformations that preserve spectral information while simplifying the counting problem. Crucially, the method distinguishes between global and local contributions, with the latter capturing the influence of each line's multiplicity.

2. Preliminaries

2.1. Multi Arrangements of Lines and Their Singularities

Let $f = \prod_{l \in L} f_l^{m_l} \in \mathbb{C}[x, y]$ be a homogeneous polynomial where each f_l is a reduced linear form. This defines:

- A singularity at the origin in $\mathbb{C}^2$
- A collection of points $\{[f_l]\}_{l \in L}$ in $\mathbb{P}^1$ corresponding to the directions of the lines
- Multiplicity m_l associated to each direction $[f_l]$

2.2. The Milnor Fiber

The local structure of the singularity is encoded in the Milnor fiber:

Definition 2.1 (Milnor Fiber). For sufficiently small $\epsilon > 0$ and $0 < |\delta| \ll \epsilon$, the Milnor fiber is:

$$M_f = \{(x, y) \in \mathbb{C}^2 : \|(x, y)\| < \epsilon,\ f(x, y) = \delta\}$$

This analytic construction provides the foundation for computing spectral invariants through its monodromy action and mixed Hodge structure.

2.3. The Spectrum

The interplay between the monodromy action and mixed Hodge structure on the vanishing cohomology yields a fundamental invariant - the spectrum - which can be expressed as a fractional Laurent polynomial:

Definition 2.2 (Spectrum). The spectrum of f is:

$$Sp_f(t) = \sum_{\alpha \in \mathbb{Q}} n_{f,\alpha} t^{\alpha}$$

where the multiplicities $n_{f,\alpha}$ are given by:

$$n_{f,\alpha} = \sum_{j \in \mathbb{Z}} (-1)^{j-1} \dim Gr_F^p \tilde{H}^j(M_f, \mathbb{C})_\lambda$$

with $p = \lfloor 2 - \alpha \rfloor$ and $\lambda = \exp(-2\pi i \alpha)$. Here:

- $\tilde{H}^j(M_f, \mathbb{C})_\lambda$ is the λ-eigenspace of the reduced cohomology under the monodromy
- F is the Hodge filtration on the cohomology
- The rational numbers α where $n_{f,\alpha} \neq 0$ are called the spectral numbers
- These spectral numbers reflect both the eigenvalues of the monodromy and the Hodge filtration

Our combinatorial approach will compute these multiplicities through lattice point enumeration.

2.4. Known Results

For multi-arrangements of lines, the spectrum can be computed using the following formula from [4].

Let $f = \prod_{l \in L} f_l^{m_l}$ be a product of linear forms, where each f_l is a distinct linear form in $\mathbb{C}[x, y]$, and m_l is the multiplicity associated with the line l. The total degree of f is $d = \sum_{l \in L} m_l$, and the reduced degree is $d_{\text{red}} = |L|$, which is the number of distinct lines in the arrangement.

We also define the Kronecker delta function $\delta_{k,d}$ as:

$$\delta_{k,d} = \begin{cases} 1 & \text{if } k = d, \\ 0 & \text{if } k \neq d. \end{cases}$$

With these definitions, the spectrum of f can be computed as follows.

Theorem 2.3 (Previous Formula). *Let $f = \prod_{l \in L} f_l^{m_l}$ be as above, with total degree d. Then, for $k \in \{1, 2, \ldots, d\}$, the spectral multiplicities are given by:*

$$n_{f,\frac{k}{d}} = d_{\text{red}} - \sum_{l \in L} \left\lceil \frac{k m_l}{d} \right\rceil + k - 1,$$

$$n_{f,1+\frac{k}{d}} = \sum_{l \in L} \left\lceil \frac{k m_l}{d} \right\rceil - k - 1 + \delta_{k,d}.$$

For all other $\alpha \in \mathbb{Q}$, we have $n_{f,\alpha} = 0$.

While this formula provides a complete description of the spectrum, its combinatorial structure is somewhat hidden behind complex expressions involving ceiling functions and sums. Our approach will reveal this structure through explicit lattice point counting in certain regions of $\mathbb{Z}^2$, making the combinatorial aspects more transparent.

3. Lattice Regions via Generating Vectors

Let $f = \prod_{l \in L} f_l^{m_l} \in \mathbb{C}[x, y]$ be a homogeneous polynomial defining a hypersurface in $\mathbb{C}^2$ with a singularity at the origin. Our approach to computing the spectrum relies on enumerating lattice points in certain regions determined by generating vectors.

3.1. Global and Local Regions

Definition 3.1. For a positive integer d (the total degree of f), we define:

1. The global region D^2 is the parallelogram generated by vectors:
$$w_1 = (d, 0), \quad w_2 = (0, d)$$
That is,
$$D^2 = \{s_1 w_1 + s_2 w_2 : 0 < s_i < 1 \text{ for } i = 1, 2\} \cap \mathbb{Z}^2$$
2. For each line L_l with multiplicity m_l, the local region $M^2_{m_l}$ is generated by:
$$v_1 = (m_l, d - m_l), \quad v_2 = (0, d)$$
That is,
$$M^2_{m_l} = \{s_1 v_1 + s_2 v_2 : 0 < s_1 < 1 \text{ and } 0 < s_2 \leq 1\} \cap \mathbb{Z}^2$$

Remark 3.2. The key features of these regions are:

- Both regions are determined by two generating vectors reflecting the structure of $\mathbb{C}^2$
- The vector $(0, d)$ appears in both regions due to the fact that each local singularity can be transformed to $x = 0$ by a change of variables
- The local regions incorporate the multiplicity m_l in their first generator
- The coefficient s_2 in the set $M^2_{m_l}$ includes 1

3.2. Weight Function and Slicing

Each coordinate provides a natural weight function $h(a_1, a_2) = \frac{a_1}{d} + \frac{a_2}{d}$ on both regions. This leads to our key definition:

Definition 3.3. For any $\alpha \in \mathbb{Q}$, we define:
$$D^2_\alpha = \{p \in D^2 : h(p) = \alpha\}$$
$$M^2_{m,\alpha} = \{p \in M^2_m : h(p) = \alpha\}$$

Remark 3.4. The key features of these regions are:

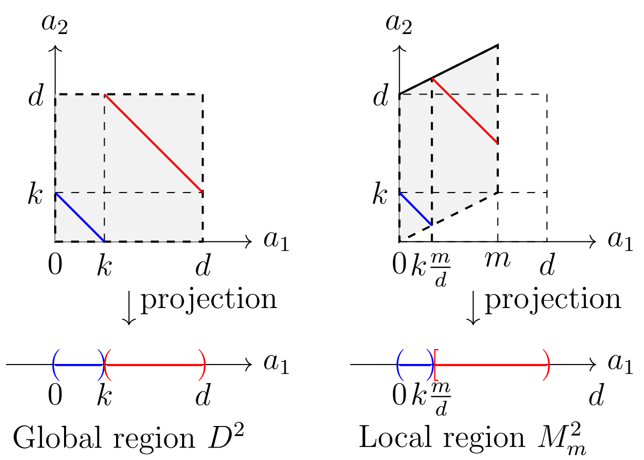

FIGURE 1. Comparison of global region D^2 and local region M^2_m. Diagonal slices (blue and red lines) and their projections onto the a_1-axis determine the spectrum contributions at weights $\frac{k}{d}$ and $1 + \frac{k}{d}$.

- Both regions are determined by two generating vectors reflecting the structure of $\mathbb{C}^2$
- The vector $(0, d)$ appears in both regions
- The local regions incorporate the multiplicity m_l in their first generator
- The coefficient s_2 in the set $M^2_{m_l}$ includes 1

PROPOSITION 3.5. *The slices exhibit the following structure:*

1. $M^2_{m_l,\alpha}$ *and* D^2_α *are subsets of* $\{(a_1, a_2) \in \mathbb{R}^2 : \frac{a_1}{d} + \frac{a_2}{d} = \alpha\}$
2. $D^2 = \cup_\alpha D^2_\alpha$ *and* $M^2_{m_l} = \cup_\alpha M^2_{m_l,\alpha}$
3. *For* $k \in \{1, \ldots, d\}$*, non-empty slices occur only at* $\alpha = \frac{k}{d}$ *and* $\alpha = 1 + \frac{k}{d}$

This slicing structure, illustrated in Figure 1, is the key to our combinatorial interpretation of the spectrum. In the next section, we will see how the multiplicities $n_{f,\alpha}$ can be computed by counting lattice points in these slices.

4. Main Results

4.1. The Spectrum via Lattice Points

Let $f = \prod_{l \in L} f_l^{m_l} \in \mathbb{C}[x, y]$ be a homogeneous polynomial defining a hypersurface in $\mathbb{C}^2$ with a singularity at the origin. Our approach to computing the spectrum relies on enumerating lattice points in certain regions determined by generating vectors. These regions arise naturally from the structure of our vector space and the line arrangement:

For a positive integer d (the total degree of f), we define:

1. The global region D^2 is the parallelogram generated by vectors:
$$w_1 = (d, 0), \quad w_2 = (0, d)$$
That is,
$$D^2 = \{s_1 w_1 + s_2 w_2 : 0 < s_i < 1 \text{ for } i = 1, 2\} \cap \mathbb{Z}^2$$
2. For each line L_l with multiplicity m_l, the local region $M_{m_l}^2$ is generated by:
$$v_1 = (m_l, d - m_l), \quad v_2 = (0, d)$$
That is,
$$M_{m_l}^2 = \{s_1 v_1 + s_2 v_2 : 0 < s_1 < 1 \text{ and } 0 < s_2 \leq 1\} \cap \mathbb{Z}^2$$

These regions capture both the global structure of the arrangement through D^2 and the local contributions of each line through $M_{m_l}^2$.

Remark 4.1. The key features of these regions are:

- Both regions are determined by two generating vectors reflecting the structure of $\mathbb{C}^2$
- The vector $(0, d)$ appears in both regions due to the fact that each local singularity can be transformed to $x = 0$ by a change of variables
- The local regions incorporate the multiplicity m_l in their first generator
- The coefficient s_2 in the set $M_{m_l}^2$ includes 1

We begin with a fundamental observation about the weight structure of these regions.

LEMMA 4.2 (Weight Structure). *For a point $p = (a_1, a_2)$ in either D^2 or $M^2_{m_l}$, the quantity $h(p) = \frac{1}{d}(a_1+a_2)$ measures its contribution to the spectrum multiplicity. In particular:*

1. *Points with $a_1 + a_2 = d$ have weight 1*
2. *The total contribution at weight α is captured by the slices D^2_α and $M^2_{m_l,\alpha}$*

Proof. For any point $p = (a_1, a_2)$ in either region, the weight $h(p) = \frac{a_1+a_2}{d}$ measures its position relative to the lines $a_1 + a_2 = kd$ for integers k.

The line $a_1 + a_2 = d$ corresponds to weight 1, as:

$$h(p) = \frac{a_1 + a_2}{d} = \frac{d}{d} = 1$$

By construction of our regions and the integrality of coordinates, lattice points can only occur at weights of the form $\frac{k}{d}$ or $1 + \frac{k}{d}$ for $k \in \{1, \dots, d\}$, and these points must lie in the corresponding slices. □

Each coordinate provides a natural weight function $h(a_1, a_2) = \frac{a_1}{d} + \frac{a_2}{d}$ on both regions. This leads to our key notion of slices:

For any $\alpha \in \mathbb{Q}$, we define:

$$D^2_\alpha = \{p \in D^2 : h(p) = \alpha\}$$
$$M^2_{m,\alpha} = \{p \in M^2_m : h(p) = \alpha\}$$

PROPOSITION 4.3. *The slices exhibit the following structure:*

1. *$M^2_{m_l,\alpha}$ and D^2_α are subsets of $\{(a_1, a_2) \in \mathbb{R}^2 : \frac{a_1}{d} + \frac{a_2}{d} = \alpha\}$*
2. *$D^2 = \cup_\alpha D^2_\alpha$ and $M^2_{m_l} = \cup_\alpha M^2_{m_l,\alpha}$*
3. *For $k \in \{1, \dots, d\}$, non-empty slices occur only at $\alpha = \frac{k}{d}$ and $\alpha = 1 + \frac{k}{d}$*

Proof. Properties (1) and (2) follow directly from the definitions of our regions and the weight function.

For (3), observe that lattice points in both regions must satisfy:

$$\frac{a_1 + a_2}{d} = \alpha$$

where a_1, a_2 are integers. Since $0 < a_1, a_2 < d$, we must have $\alpha \in (\frac{k}{d}, 1+\frac{k}{d})$ for some $k \in \{1, \ldots, d\}$. Furthermore, the integrality of coordinates forces α to take these specific values. □

This slice structure leads to our main theorem, which expresses the spectrum through lattice point enumeration:

THEOREM 4.4 (Main Theorem). *Let $f = \prod_{l \in L} f_l^{m_l}$ be a homogeneous polynomial with degree d. Then for any $\alpha \in \mathbb{Q}$:*

$$n_{f,\alpha} = |D^2_\alpha| - \sum_{l \in L} |M^2_{m_l,\alpha}|$$

where:

- *$|D^2_\alpha|$ counts lattice points in the global slice at weight α*
- *$|M^2_{m_l,\alpha}|$ counts lattice points in the local slice for the line l at weight α*
- *The sum is taken over all lines $l \in L$*

Proof. First, we show that $n_{f,\alpha} = 0$ unless $\alpha = \frac{k}{d}$ or $1 + \frac{k}{d}$ for some $k \in \{1, \ldots, d\}$. This follows from Lemma 4.2, as all lattice points lie on lines $a_1 + a_2 = kd$.

For these potential non-zero values, we establish the counting formula by showing it yields the known multiplicities:

For $\alpha = \frac{k}{d}$:

$$\begin{aligned} n_{f,\frac{k}{d}} &= |L| - \sum_{l \in L} \lceil k m_l/d \rceil + k - 1 \\ &= (k-1) - \sum_{l \in L} (\lceil k m_l/d \rceil - 1) \end{aligned}$$

For $\alpha = 1 + \frac{k}{d}$:

$$\begin{aligned} n_{f,1+\frac{k}{d}} &= \sum_{l \in L} \lceil k m_l/d \rceil - k - 1 + \delta_{k,d} \\ &= -\left(d - \sum_{l \in L} \lceil k m_l/d \rceil \right) + d - k - 1 + \delta_{k,d} \end{aligned}$$

These exact values match the classical formulas from Theorem 2.3, completing the proof. □

This geometric interpretation leads to fundamental properties of the spectrum:

Corollary 4.5 (Structural Properties). *For a line arrangement with multiplicities:*

1. *The spectrum is supported in the interval* $(0, 2)$
2. *While the global contribution* $|D^2_\alpha|$ *exhibits perfect symmetry about* $\alpha = 1$*, the local contributions* $|M^2_{m_l,\alpha}|$ *generally do not*
3. *The spectrum exhibits symmetry around* $\alpha = 1$ *if and only if the singularity is isolated*

Proof. (1) The support property follows from our weight function $h(a_1, a_2) = \frac{a_1+a_2}{d}$ taking values in $(0, 2)$ on both regions, as each coordinate is bounded between 0 and d.

(2) The global region D^2 is symmetric about the line $a_1 + a_2 = d$, which corresponds to weight 1. This symmetry induces a bijection between lattice points at weights α and $2 - \alpha$. However, the local regions $M^2_{m_l}$ include the face where $s_2 = 1$, breaking this symmetry.

(3) The symmetry equivalence follows because the global contribution is always symmetric, while local contributions break this symmetry. Thus, symmetry occurs if and only if there are no local contributions, which happens precisely when the singularity is isolated. □

4.2. Computation of Spectral Numbers

To make our main theorem effective, we need explicit methods for counting lattice points. We begin with some specialized counting functions:

Lemma 4.6 (Counting Functions). *For any real number* β *and positive integer* a*, define:*

$$u_a(\beta) = \begin{cases} \lceil \beta \rceil - 1 & \text{if } 0 < \beta \leq a \\ 0 & \text{otherwise} \end{cases}$$

and

$$v_a(\beta) = \begin{cases} a - \lceil \beta \rceil & \text{if } 0 < \beta < a \\ 0 & \text{otherwise} \end{cases}$$

Then:

1. $u_a(\beta)$ *counts integers in* $(0, \beta)$
2. $v_a(\beta)$ *counts integers in* $[\beta, a)$
3. $u_a(\beta) + v_a(\beta) = a - 1$ *when* $0 < \beta < a$

Proof. Properties (1) and (2) follow directly from the definitions. For (1), when $0 < \beta \leq a$, $u_a(\beta)$ counts integers n satisfying $0 < n < \beta$, which is exactly $\lceil \beta \rceil - 1$. When $\beta \leq 0$ or $\beta > a$, no such integers exist.

Similarly for (2), when $0 < \beta < a$, $v_a(\beta)$ counts integers n satisfying $\beta \leq n < a$, which is exactly $a - \lceil \beta \rceil$. When $\beta \leq 0$ or $\beta \geq a$, either all or no integers in $[1, a)$ satisfy the condition.

For (3), observe that when $0 < \beta < a$, the intervals $(0, \beta)$ and $[\beta, a)$ partition the integers in $(0, a)$, of which there are $a - 1$ many. □

These counting functions allow us to give explicit formulas for the lattice point counts:

THEOREM 4.7 (Explicit Enumeration). *For* $k \in \{1, \dots, d\}$*:*

1. *For the global region:*

$$|D^2_{\frac{k}{d}}| = k - 1 \quad \text{and} \quad |D^2_{1+\frac{k}{d}}| = d - k - 1$$

2. *For each local region:*

$$|M^2_{m_l, \frac{k}{d}}| = \lceil k m_l / d \rceil - 1$$
$$|M^2_{m_l, 1+\frac{k}{d}}| = m_l - \lceil k m_l / d \rceil$$

Proof. For the global region D^2, at weight $\frac{k}{d}$, we are counting lattice points (a_1, a_2) satisfying:

$$\frac{a_1 + a_2}{d} = \frac{k}{d}, \quad 0 < a_1, a_2 < d$$

This reduces to counting positive integer solutions to $a_1 + a_2 = k$ with each a_i less than k, giving $k - 1$ points.

For weight $1 + \frac{k}{d}$, we use the complementary count relative to d and the symmetry of D^2.

The local region counts follow similarly, using the counting functions u_a and v_a with appropriate scaling by $\frac{m_l}{d}$. The shape of the local regions $M^2_{m_l}$ ensures the lattice points occur exactly where predicted by these counting functions. □

Remark 4.8. These explicit formulas directly connect our geometric perspective with classical results, showing how the ceiling function $\lceil\cdot\rceil$ arises naturally from lattice point positions relative to diagonal slices.

5. The Monomial Case

We now demonstrate the power of our approach by analyzing the monomial case $f(x,y) = x^{m_1}y^{m_2}$ in detail. This case, while simple, illuminates the key features of our method and provides combinatorial insight into classical formulas.

Example 5.1. Consider $f(x,y) = x^6y^9$. We have:

- Global region D^2: A square from $(0,0)$ to $(15,15)$
- Two local regions:
  - M_6^2 for the line $x = 0$ with multiplicity 6
  - M_9^2 for the line $y = 0$ with multiplicity 9

For weight $\alpha = \frac{5}{18}$:

$$|D^2_{\frac{5}{18}}| = 1 \text{ (lattice point at } (1,1))$$
$$|M^2_{6,\frac{5}{18}}| = 0 \text{ (no lattice points)}$$
$$|M^2_{9,\frac{5}{18}}| = 1 \text{ (lattice point at } (1,1))$$

Therefore $n_{f,\frac{2}{5}} = 0$. Similar computations give the full spectrum:

$$Sp_f(t) = t^{\frac{1}{3}} + t^{\frac{2}{3}} + t - t^{1+\frac{1}{3}} - t^{1+\frac{2}{3}}$$

For the general case $f(x,y) = x^{m_1}y^{m_2}$, our method reveals the deep connection between the geometry of the regions and arithmetic properties of the multiplicities:

Theorem 5.2. *Let $m := \gcd(m_1, m_2)$ and define $m_1' := \frac{m_1}{m}$, $m_2' := \frac{m_2}{m}$. Then:*

1. *The spectral multiplicities are 1 for $\alpha \leq 1$ and -1 for $\alpha > 1$*
2. *On the line segment from $(0,0)$ to (m_1, m_2):*
   - *Lattice points occur at $k(m_1', m_2')$ for $k = 1, \ldots, m$*
   - *These points contribute to weights $\frac{k}{m}$ for $k = 1, \ldots, m$*
3. *On the line segment from (m_1, m_2) to (d, d):*

- *Lattice points occur at* $(m_1, m_2) + k(m'_2, m'_1)$ *for* $k = 1, \ldots, m-1$
- *These points contribute to weights* $1+\frac{k}{m}$ *for* $k = 1, \ldots, m-1$

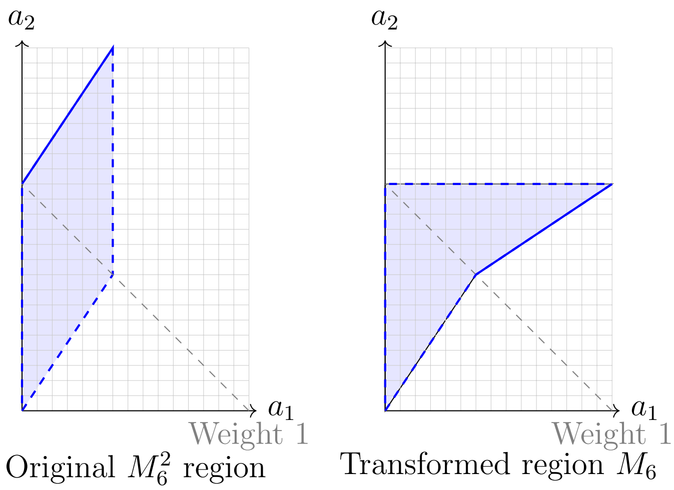

FIGURE 2. Weight-preserving transformation applied to the local region M_6^2. Left: original region. Right: transformed region with preserved lattice point weights.

Proof. The key insight is that both local regions $M_{m_1}^2$ and $M_{m_2}^2$ can be transformed while preserving lattice points and weights. Specifically:

1. For $M_{m_1}^2$, we transform the region while preserving lattice point weights relative to the line $a_1 + a_2 = d$
2. For $M_{m_2}^2$, we apply a two-step transformation to minimize overlap with the transformed $M_{m_1}^2$ region
3. The intersection points of the transformed regions precisely capture the role of $\gcd(m_1, m_2)$

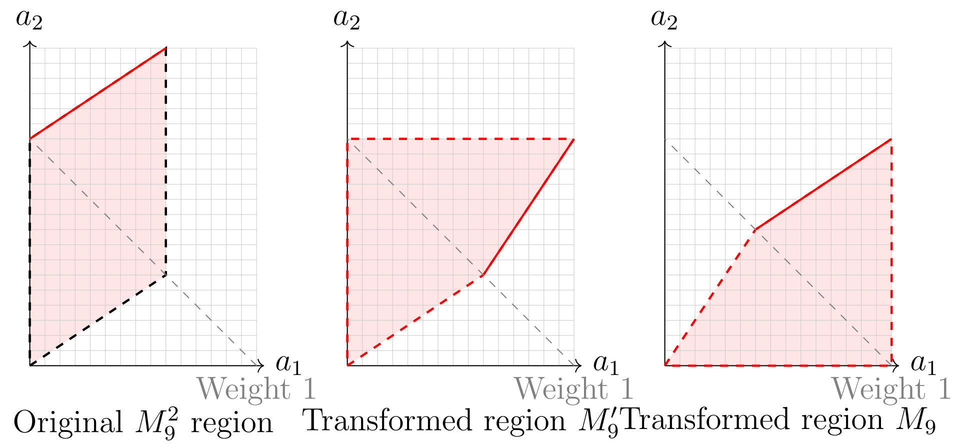

FIGURE 3. Two-step transformation of M_9^2 showing the original region (left), intermediate step (center), and final configuration (right).

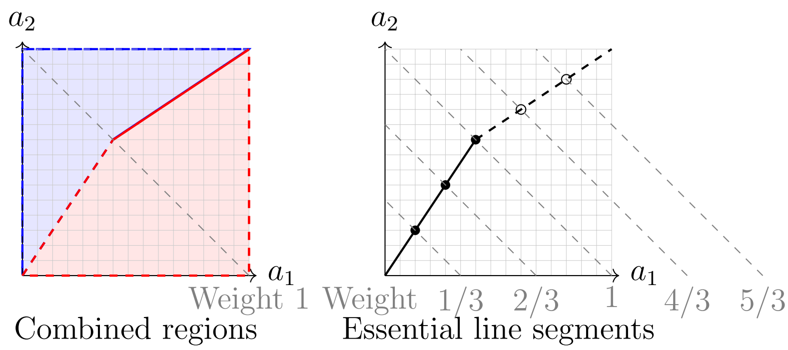

FIGURE 4. Left: Intersection of transformed local regions. Right: The resulting essential line segments where all relevant lattice points must lie.

The lattice points on the resulting line segments then determine the spectrum through:

$$Sp_f(t) = t + (1-t)\frac{t^{\frac{1}{m}} - t}{1 - t^{\frac{1}{m}}} = t^{\frac{1}{m}} + t^{\frac{2}{m}} + \cdots + t^1 - t^{1+\frac{1}{m}} - t^{1+\frac{2}{m}} - \cdots - t^{2-\frac{1}{m}}$$

□

Remark 5.3. This combinatorial approach provides natural explanations for:

1. The appearance of $\gcd(m_1, m_2)$ through lattice enumeration
2. Symmetry breaking between weights less than 1 and greater than 1
3. The role of primitive vectors (m_1', m_2') in determining spectral values

6. Conclusion and Future Work

In this paper, we introduced a combinatorial method for computing the spectrum of singularities using lattice point enumeration in regions determined by the defining equation of line arrangements with multiplicities. Our approach provides explicit formulas for spectral multiplicities and reveals the interplay between global and local contributions to the spectrum through the counting of lattice points.

Specifically, we demonstrated how transformations of these regions preserve the spectral data while simplifying the counting process. The method was applied to the monomial case $f(x, y) = x^{m_1}y^{m_2}$, offering a natural explanation for the role of $\gcd(m_1, m_2)$ in the spectrum. This work provides a new perspective on the spectra of line arrangements with multiplicities and enhances the understanding of their combinatorial structures.

Future research may focus on extending this combinatorial approach to other classes of non-isolated singularities. Investigating the applicability of our method to more general hypersurfaces or exploring the connections between lattice point enumeration and other invariants in singularity theory could provide valuable insights. Additionally, refining the counting techniques or developing computational tools to facilitate these calculations may further advance the study of spectra in complex geometry.

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